

Long-range interactions in the quantum many-body problem in one dimension: Ground state

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We investigate the ground state properties of a family of N -body systems in one dimension, trapped in a polynomial potential and having long-range two-body interaction in addition to the inverse square potential studied in the Calogero-Sutherland model (CSM). We show that for such a Hamiltonian, the ground state energy is similar to that of free fermions in a harmonic well with a displacement that depends on the number of particles and depth of the well. We obtain the ground state wave function and using random matrix results, study the particle density and pair correlation function (PCF). We observe that the particles are arranged in bands. Due to the presence of long-range interaction, the PCF shows a departure from the CSM.

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Theoretical understanding of the ground state properties of complex many-body systems has received considerable attention in recent years [1–13]. In this context, we study rigorously a wide class of one-dimensional N -body systems having different densities, nature, and strength of a two-body interaction. We obtain the ground state properties of these systems having a two-body potential $V_2 = g/r_{12}^2 + \Phi_l$, where r_{12} is the interparticle spacing and Φ_l contains the long-range two-body interaction. The system is trapped in a polynomial potential. This may be relevant in understanding the various aspects of the Bose-Einstein condensates, where a wide variety of potentials under a controlled environment is possible.

We derive the ground state eigenvalues and eigenfunctions for such systems and extract several interesting properties by identifying the square of the wave function with the joint probability distribution (JPD) of eigenvalues of non-Gaussian ensembles of random matrices. Using the polynomial method developed by Ghosh and Pandey in the context of random matrix theory (RMT) [14,15], we observe band structure [16] in the particle density, which in turn corresponds to the density of zeros of the corresponding polynomials [7,14,15]. For a given value of the interaction strength, we study the pair correlation function (PCF) for different interparticle spacings. Due to the presence of the long-range interaction, we observe a deviation from the Calogero-Sutherland model (CSM) [1,2].

We shall consider the ground state of a system of N particles satisfying the Schrödinger equation

$$\left[-\sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + \prod_{i<j} \frac{g}{(x_i - x_j)^2} + \Phi_l(x_i, x_j) + V_1(x_i) - E_n \right] \psi_n = 0, \quad (1)$$

where terms corresponding to long-range interaction derive from the relation

$$\Phi_l(x_i, x_j) = h \sum_{i \neq j} \frac{P(x_i)}{(x_i - x_j)}, \quad (2)$$

$P(x)$ being any analytic function having a power expansion in x , while g and h are the interaction strengths. The system is trapped in a potential

$$V_1(x_i) = \sum_{i=1}^N [P^2(x_i) - P'(x_i)]. \quad (3)$$

In this paper, we consider the case where $P(x)$ is a polynomial of order $2m+1$ and is represented by

$$P(x) = \gamma \sum_{k=0}^m a_{2k+1} x^{2k+1}. \quad (4)$$

For convenience, we take $a_{2m+1} = 1$. The parameter γ will determine the depth of the well. The case where $m=0$ corresponds to the CSM [1], where the two-body interaction is purely of the inverse square type. For $m>0$, we will encounter the long-range interaction.

For such a Hamiltonian, the ground state wave function can be written as

$$\psi_0 \equiv \phi \varphi = \prod_{i<j} |x_i - x_j|^\lambda \exp \left[-\sum_i \int_0^{x_i} P(t_i) dt_i \right]. \quad (5)$$

It should be noted that if the particles are confined to a configuration space $x_1 > x_2 > \dots > x_N$, the g/r^2 interaction in one dimension does not allow them to cross, thereby respecting the ordering. This is valid provided the potential is not too attractive, which leads to the restriction $g \geq -1/2$ [1,17]. For such an ordering, the modulus becomes irrelevant and hence depending on the value of λ (odd or even) the system is bosonic or fermionic in nature. Also, for such a configuration, ψ_0 is nodeless (apart from the trivial ones at the points of coincidence) and hence corresponds to the ground state of the system. Here

$$\phi = \prod_{i<j} |x_i - x_j|^\lambda, \quad (6)$$

$$\varphi = \exp \left[-\sum_i \int_0^{x_i} P(t_i) dt_i \right], \quad (7)$$

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$$(\lambda^2 - \lambda) = g/2, \quad (8)$$

and

$$2\lambda = -h. \quad (9)$$

The ground state energy is given by

$$E_0 = -\gamma a_1 N [1 + \lambda(N-1)]. \quad (10)$$

For $a_1 > 0$, this is the same as that of free fermions in a harmonic well γa_1 , displaced by a factor $\gamma \lambda N a_1 (N-1)$.

First we prove that for $P(x_i)$ given by Eq. (4), Eqs. (5) and (10) are, respectively, the ground state eigenfunction and eigenvalue. We study two cases, where we calculate ψ_0 and E_0 and recognize ψ_0^2 as the JPD of eigenvalues for the non-Gaussian ensembles of random matrices. Using the polynomial method, we study the particle density and the effect of long-range interaction on the PCF. PCF shows a deviation from the CSM, which thereby suggests a departure from the ‘‘universal’’ result of RMT.

The kinetic term of the Hamiltonian acting on ψ_0 gives

$$\begin{aligned} -\sum \frac{\partial^2(\phi\varphi)}{\partial x_i^2} &= -\sum \varphi \frac{\partial^2(\phi)}{\partial x_i^2} - 2\sum \frac{\partial(\varphi)}{\partial x_i} \frac{\partial(\phi)}{\partial x_i} \\ &\quad - \sum \phi \frac{\partial^2(\varphi)}{\partial x_i^2}. \end{aligned} \quad (11)$$

Substituting ψ_0 from Eq. (5), the first term gives

$$\sum \varphi \frac{\partial^2(\phi)}{\partial x_i^2} = \left[2(\lambda^2 - \lambda) \sum_{i < j} \frac{1}{(x_i - x_j)^2} \right] \phi\varphi. \quad (12)$$

The second term gives

$$-2\sum \frac{\partial(\varphi)}{\partial x_i} \frac{\partial(\phi)}{\partial x_i} = 2\lambda \phi\varphi \sum_{i \neq j} \frac{P(x_i)}{(x_i - x_j)}, \quad (13)$$

while it is easy to see that the last term

$$-\sum \phi \frac{\partial^2(\varphi)}{\partial x_i^2} = \sum_i [P'(x_i) - P^2(x_i)] \phi\varphi. \quad (14)$$

Thus we see that our choice of ψ_0 diagonalizes the Hamiltonian (1). Replacing $P(x_i)$ from Eq. (4) in Eqs. (13) and (14), one can easily obtain the ground state energy for the cases where $m \geq 0$.

We will consider the two cases, corresponding to

$$P(x_i) = \gamma(x_i^3 - a_1 x_i), \quad \gamma > 0 \quad (15)$$

and

$$P(x_i) = \gamma(x_i^5 - a_3 x_i^3 + a_1 x_i), \quad \gamma > 0, \quad a_3 < 0. \quad (16)$$

They will not only illustrate the formation of multiple bands in the particle density for appropriate values of the parameter a_1 , but also show the effect of long range interaction in determining the PCF.

The Hamiltonian corresponding to Eq. (15) is

$$\begin{aligned} H &= -\sum \frac{\partial^2}{\partial x_i^2} + \prod_{i < j} \frac{g}{(x_i - x_j)^2} - \frac{h\gamma}{2} \sum_{i < j} (x_i - x_j)^2 \\ &\quad + \gamma^2 \sum_{k=1}^3 b_{2k} x_i^{2k}. \end{aligned} \quad (17)$$

For $1 \gg \gamma > 0$, the system behaves like an anisotropic oscillator, kept in a weak polynomial well. For such systems, the motion is bounded for all physically possible energies. In the thermodynamic limit, i.e., for $N \rightarrow \infty$ and $\gamma \rightarrow 0$, we will have a finite particle density, whose shape will depend on the well it is kept in. Finally, repeating the steps outlined earlier, one can find that a choice of ψ_0 ,

$$\psi_0 \equiv \phi\varphi = \prod_{i < j} |x_i - x_j|^\lambda \exp \left[-\gamma \left(\sum_i \frac{x_i^4}{4} - a_1 \frac{x_i^2}{2} \right) \right], \quad \gamma > 0, \quad (18)$$

satisfies the Schrödinger equation, with g and h given by Eqs. (8) and (9), respectively, and

$$b_6 = 1, \quad b_4 = -2a_1, \quad b_2 = a_1^2 - (6\lambda + 3)/\gamma. \quad (19)$$

For $\gamma a_1^2 = 6\lambda + 3$ and $\gamma \rightarrow 0$, we can obtain the ground state wave function and eigenvalue for the central potential $V(r_{12}) = A r_{12}^2 + B/r_{12}^2$ and compare the result for $N=2$ with that obtained in Ref. [17]. For such a choice, we find $E_0 = 2(1 + \lambda) \sqrt{[3\gamma(2\lambda + 1)]}$ as compared to $E_0 = \sqrt{\lambda\gamma(2\lambda + 1)}$ obtained in Ref. [17]. The deviation is due to the effect of the potential well. For N particles, the ground state energy is given by Eq. (10) with a negative sign introduced due to the negative coefficient a_1 . The Hamiltonian corresponding to Eq. (16) can be written as

$$\begin{aligned} H &= -\sum \frac{\partial^2}{\partial x_i^2} + \prod_{i < j} \frac{g}{(x_i - x_j)^2} + \alpha_1 \sum_{i < j} (x_i - x_j)^4 \\ &\quad + \alpha_2 \sum_{i < j} (x_i - x_j)^2 + \alpha_3 \sum_{i < j} (x_i + x_j)^4 + \gamma^2 \sum_{k=1}^5 b_{2k} x_i^{2k}. \end{aligned} \quad (20)$$

Here, the second, third, and fourth terms have the same effect of making the particles remain in a bound state, with the parameter γ and a_3 controlling, respectively, the range and depth of the potential.

Repeating the steps outlined earlier, one can find that a choice of ψ_0 ,

$$\begin{aligned} \psi_0 &= \prod_{i < j} |x_i - x_j|^\lambda \exp \left[-\gamma \left(\sum_i \frac{x_i^6}{6} - a_3 \sum_i \frac{x_i^4}{4} + a_1 \frac{x_i^2}{2} \right) \right], \\ &\quad \gamma > 0, \end{aligned} \quad (21)$$

with $\gamma > 0$, $a_3 < 0$ satisfies the Schrödinger equation, with h given by Eq. (9), and

$$\alpha_1 = \lambda \gamma / 12, \quad \alpha_2 = -\lambda \gamma a_3, \quad \alpha_3 = -5\lambda \gamma / 12,$$

$$b_{10} = 1, \quad b_8 = -2a_3, \quad b_6 = (a_3^2 + 2a_1),$$

$$b_4 = -[2a_1 a_3 + (5 + 10\lambda/3)/\gamma], \quad b_2 = a_1^2 + a_3(6\lambda + 3)/\gamma. \quad (22)$$

The ground state energy is given by Eq. (10). As in the previous case, the parameter a_1 will not only determine the position of the ground state but will also be crucial in determining the band structure.

It is at this point that we will write ψ_0 in terms of the variables

$$y_i = \left(\frac{1}{\lambda}\right)^{1/(2m+2)} x_i. \quad (23)$$

Then ψ_0 can be written as

$$\psi_0 = C^{1/2} \prod_{i < j} |y_i - y_j|^\lambda \exp\left[-\lambda \sum_i \int_0^{y_i} P_1(t_i) dt_i\right], \quad (24)$$

where

$$P_1(y_i) = \gamma \sum_{k=0}^m c_{2k+1} y_i^{2k+1} \quad (25)$$

is a monic polynomial of order $2m + 1$, whose coefficients are related to those of $P(x_i)$ by

$$c_{2k+1} = \left(\frac{1}{\lambda}\right)^{(m-k)/(m+1)} a_{2k+1}. \quad (26)$$

Then ψ_0^2 is given by

$$\psi_0^2 = C \prod_{i < j} |y_i - y_j|^\beta \exp\left[-\beta \sum_i \int_0^{y_i} P_1(t_i) dt_i\right]. \quad (27)$$

with $\beta = 2\lambda$, and C being the normalization constant. Now one may interpret ψ_0^2 to be identical with the JPD of non-Gaussian ensembles of random matrices [15]. We define the n -particle correlation function

$$R_n(y_1, y_2, \dots, y_n) = \int \dots \int dy_{n+1} \dots dy_N \psi_0^2 \quad (28)$$

as the probability of finding n particles in the intervals y_i and $y_i + \Delta y_i$, irrespective of the position of the other particles. $n = 1$ and 2 correspond to the particle density and PCF, respectively. It is shown in Ref. [15] that $R_1(x)$ corresponds to the density of zeros of the polynomial having weight function given by Eq. (7). It has been shown by Dyson in the context of random matrices that for $\beta = 1, 2$, and 4 , R_n can be written in terms of orthogonal and skew-orthogonal polynomials. For $\beta = 2$ (Fig. 1), the PCF can be written as

$$R_2(y_1, y_2) = \sum_{\mu=0}^{N-1} h_\mu^{-1} [q_\mu(y_1) q_\mu(y_2)] \exp\left[-2 \int_0^{y_2} P_1(t) dt\right], \quad (29)$$

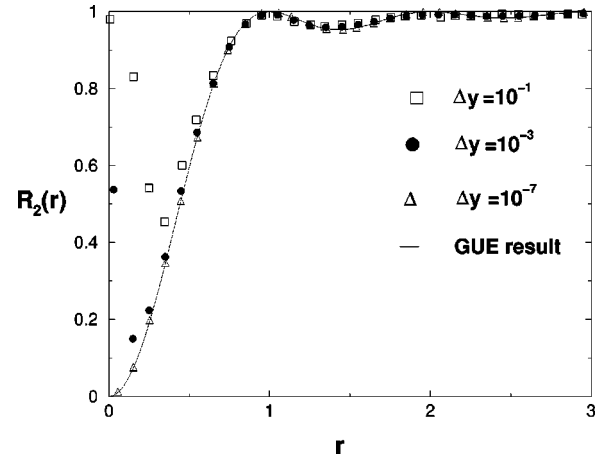


FIG. 1. It shows the smoothed PCF [15] for the Hamiltonian (17), with $\beta = 2$, $N = 50$, $a_1 = 60$, and $r = \Delta y R_1(y)$. The curves correspond to different values of Δy . The solid line corresponds to CSM (GUE) result, with $\Delta y = 10^{-2}$.

where $q_\mu(y)$ are orthogonal polynomials corresponding to the normalization condition

$$\int_{-\infty}^{\infty} q_\mu(y) q_\nu(y) \exp\left[-2 \int_0^y P_1(t) dt\right] dy = h_\mu \delta_{\mu\nu}. \quad (30)$$

Now, we come back to the Hamiltonian discussed in Eq. (17). For $a_1 > 0$, the ground state energy is positive. For $a_1 \equiv a_c = \sqrt{2N/\gamma}$, we define $E_{0c} = \gamma a_c N [1 + \lambda(N - 1)]$. For $E_0 > E_{0c}$, we observe the formation of two bands in particle density. Rescaling the result obtained by Pandey [16], we get

$$\pi R_1(y) = \gamma |y| \sqrt{2N/\gamma - (y^2 - a_1)^2}. \quad (31)$$

For $-E_{0c} < E_0 < 0$, a passage through the barrier is possible resulting in a splitting of each of these levels into two neighboring ones, corresponding to the state in which the particles move simultaneously through both the barriers. This corresponds to a single-band case with a dip around the origin. Finally, for $E_0 < -E_{0c}$, we get a single band with a maximum at $y = 0$. This is given by

$$\pi R_1(y) = \gamma \left[\frac{1}{3\gamma} \{ \sqrt{a_1^2 + 6N/\gamma} - 2a_1 \} + y^2 \right] \times \left[\frac{2}{3\gamma} \{ \sqrt{a_1^2 + 6N/\gamma} + a_1 \} - y^2 \right]^{1/2}, \quad (32)$$

for $a_1 \leq \sqrt{2N/\gamma}$ for the single-band case. We observe that for finite $R_1(y)$, letting $N \rightarrow \infty$ implies that $\gamma \rightarrow 0$ as N^{-1} . It should be noted that in the thermodynamic limit, the particle density, after proper scaling, is independent of β and hence applicable for both bosons and fermions. However the calculation of the PCF, essential for the study of thermodynamic properties, is restricted to $\beta = 1, 2$, and 4 . It has been proved in Refs. [14,15] that for interparticle spacing $(y_1 - y_2) \equiv \Delta y \rightarrow 0$, $N \rightarrow \infty$ and defining $r = \Delta y R_1(y)$, the scaled PCF for CSM is universal. For $\beta = 2$, it is given by

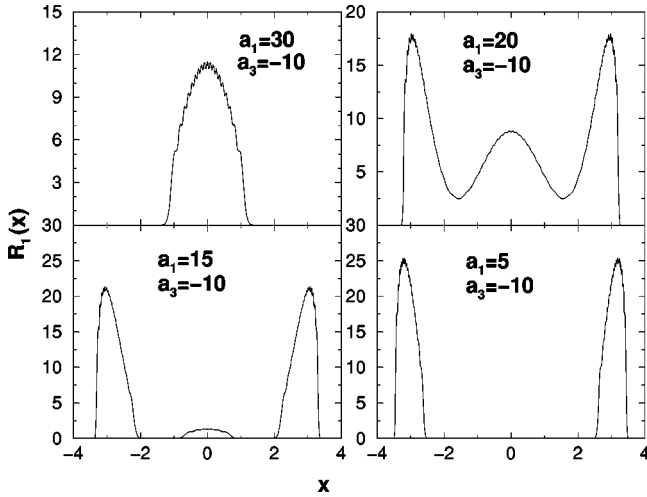


FIG. 2. Particle density for the Hamiltonian given in Eq. (20), with $N=50$. For a given a_3 and different values of the parameter a_1 , we observe a shift from single band to multiple band structure.

$$R_2(r) = 1 - \frac{\sin^2(\pi r)}{(\pi r)^2}. \quad (33)$$

For these systems with $\Delta y = 10^{-7}$, we are in the region of the spectrum where the particle density is high and hence interparticle spacing low. Here, PCF obeys Eq. (33). However, for $\Delta y = 10^{-3}$ and 10^{-1} , we are in the tail region of the two bands. It is here that we observe a distinct deviation from that observed in CSM. This is due to the contribution of the long-range interaction, where the third term in Eq. (17) contributes to the PCF. This becomes negligible as one makes Δy smaller, until it disappears for $\Delta y = 10^{-7}$.

For the Hamiltonian corresponding to Eq. (20), it can be shown analytically that for large N , the particle density (Fig.

2) can be written in terms of the higher moments as

$$\begin{aligned} \pi R_1(x) = & \gamma[-x^{10} + 2a_3x^8 - (a_3^2 + 2a_1)x^6 \\ & + (2N/\gamma + 2a_1a_3)x^4 - (a_1^2 - 2M_2/\gamma + 2a_3N/\gamma)x^2 \\ & + (2a_1N + 2M_4 - 2a_3M_2)/\gamma]^{1/2} \end{aligned} \quad (34)$$

where $M_p = \int x^p R_1(x) dx$, x being the scaled variable. These moments can be calculated numerically to obtain $R_1(x)$. However, a more convenient way is to directly construct the polynomials through the recursion relation and to do the sum of Eq. (29), with $x=y$. For $a_3 < 0$, we define $a_1 \equiv a_c = a_3^2/4$, which corresponds to a ground state energy $E_{0c} = -\gamma a_c N[1 + \lambda(N-1)]$. For $E_0 < E_{0c}$, we observe a single band structure in the particle density. For $E_0 > E_{0c}$ (i.e., $a_1 < a_c$), the particles experience a repulsion at the center and thus we see a transition from single band to three band structure. As $E_0 \ll E_{0c}$, particles are completely repelled from the center and finally collect at the two end of the spectrum, thereby giving rise to two bands.

Thus we have studied, for both bosons and fermions, the ground state properties of a class of N -body systems with long-range two-body interaction in addition to the inverse square potential. We observe a distinct deviation from the universal result for the PCF, as obtained in the CSM. We also observe that particles are arranged in bands. To study different thermodynamic properties, it is necessary to study the excitation spectrum of such Hamiltonian. At finite temperature and finite N , we expect similar smoothing of the oscillations in the particle density [16] as observed in Ref. [18]. We will come back to this in a later publication.

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